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On Müntz Rational Approximation Rate in L^p Spaces[1]

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Let $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Given $M>0$, under the hypothesis $0\leqslant\lambda_1<\lambda_2<\cdots$ and $\lambda_{n+1}-\lambda_n\geqslant Mn$, $n=1, 2, 3, \ldots$, the present paper obtains the rational Müntz approximation rate in $L^p_{[0,1]}$ space

$$R_n(f,\Lambda)_{L^p_{[0,1]}}\leqslant C_M\omega(f,n^{-1})_{L^p_{[0,1]}}$$

for $1\leqslant p<\infty$, where C_M is a positive constant only depending upon M.

1. INTRODUCTION

Let $L^p_{[0,1]}$ be the space of all p-power integrable functions on $[0,1]$, $1\leqslant p<\infty$, and $C_{[0,1]}=L^{\infty}_{[0,1]}$, for convenience, the space of all continuous functions on $[0,1]$. Given a nonnegative (strict) increasing sequence $\{\lambda_n\}$, denote by $\Pi_n(\Lambda)$ the set of Müntz polynomials of degree n, that is, all linear combinations of $\{x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}\}$, and by $R_n(\Lambda)$ the Müntz rational functions of degree n, that is,[2]

$$R_n(\Lambda)=\{P(x)/Q(x) : P, Q\in\Pi_n(\Lambda);\ Q(x)\geqslant 0,\ x\in(0,1]\}.$$

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[2] If $Q(0)=0$, we require that $\lim_{x\to 0+} P(x)/Q(x)$ exist and be finite.

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For $f \in L^p_{[0,1]}$, $1 \leqslant p \leqslant \infty$, define

$$R_n(f)_{L^p} = \inf_{r \in R_n(\Lambda)} \|f - r\|_{L^p},$$

$$\omega(f, t)_{L^p} = \sup_{|h| \leqslant t} \left(\int_E |f(x+h) - f(x)|^p \, dx \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

where $E = [0, 1-h]$ for $0 \leqslant h \leqslant 1$, or $E = [-h, 1]$ for $-1 \leqslant h < 0$, and

$$\omega(f, t)_{L^\infty} = \sup_{|h| \leqslant t} \max_{0 \leqslant x,\, x+h \leqslant 1} |f(x+h) - f(x)|.$$

As we know, it is a hard subject how to estimate general Müntz rational approximation rate. In the past 12 years, there was some nice work done in [1–3, 5, 6], and a very hard open problem left in [3]. Among them, we cite a result of Bak [1] here.

THEOREM 1. *Given $M > 0$. If $\lambda_{n+1} - \lambda_n \geqslant Mn$ for all $n \geqslant 1$, then there is a positive constant C_M only depending upon M such that*

$$R_n(f)_{L^\infty} \leqslant C_M \omega(f, n^{-1})_{L^\infty}.$$

The present paper considers generalizing the above theorem to include the general L^p spaces. In the Müntz rational approximation case, as one can see from the following proof, this work is not as easy as the usual polynomial approximation case.

We establish the following theorem.

THEOREM 2. *Given $M > 0$, $1 \leqslant p \leqslant \infty$. If $\lambda_{n+1} - \lambda_n \geqslant Mn$ for all $n \geqslant 1$, then there is a positive constant C_M only depending upon M (independent of p!) such that*

$$R_n(f)_{L^p} \leqslant C_M \omega(f, n^{-1})_{L^p}.$$

2. AUXILIARY LEMMAS

For convenience, we always write C as a positive constant that may depend upon M in various situations although different values may be assigned[3]. Also, we always assume that $\lambda_{n+1} - \lambda_n \geqslant Mn$ for all $n \geqslant 1$ in the following and $f \in L^p_{[0,1]}$ for $1 \leqslant p < \infty$.

[3] We note that C is always independent of p!.

Following Bak's idea, we set

$$P_j(x) = P_{j,n}(x) = x^{\lambda_j} \prod_{l=1}^{j} x_l^{-\Delta\lambda_l}$$

for $j = 1, 2, ..., n$ and $n = 1, 2, ...,$ where $\Delta\lambda_1 = \lambda_1$, $\Delta\lambda_n = \lambda_n - \lambda_{n-1}$, $n = 2,$ $3, ...,$ and $x_j = x_{j,n} = j/n$, $j = 0, 1, ..., n$. Furthermore, set

$$r_k(x) = \frac{P_k(x)}{\sum_{l=1}^{n} P_l(x)} \tag{1}$$

for $k = 1, 2, ..., n$. First, we need an estimate for $r_k(x)$.

LEMMA 1. *For any* $x \in [0, 1]$, $N = [nx]$, *we have, for* $k = 1, 2, ..., n$,

$$r_k(x) \leqslant C \exp(-M\,|N-k|).$$

One can find the above estimate in the proof of the theorem in Bak [1]. We also need a fundamental result in classical analysis.

LEMMA 2. *If* $f \in L^p_{[0,1]}$, $1 \leqslant p < \infty$, *and* $h > 0$, *then*

$$\int_{1-h}^{1} \int_{1-h}^{1} |f(x) - f(y)|^p \, dx \, dy \leqslant Ch\omega^p(f, h)_{L^p}.$$

Proof. Direct calculations lead to

$$\begin{aligned}
&\int_{1-h}^{1} \int_{1-h}^{1} |f(x) - f(y)|^p \, dx \, dy \\
&\quad = 2 \int_{1-h}^{1} \int_{1-h}^{y} |f(x) - f(y)|^p \, dx \, dy \\
&\quad = 2 \int_{1-h}^{1} \int_{1-h-y}^{0} |f(u+y) - f(y)|^p \, du \, dy \\
&\quad = 2 \int_{-h}^{0} \int_{1-h-x}^{1} |f(y+u) - f(y)|^p \, dy \, du \leqslant 2h\omega^p(f, h)_{L^p}. \quad \blacksquare
\end{aligned}$$

For $f \in L^p_{[0,1]}$, define now

$$\nabla_n(f, x) = \sum_{k=1}^{n} n r_k(x) \int_{(k-1)/n}^{k/n} f(u) \, du.$$

Evidently, $\nabla_n(f, x) \in R_n(\Lambda)$. We have the following lemma.

LEMMA 3. *We estimate the following*:

$$\int_{1-1/n}^{1} |f(x)-\nabla_n(f, x)|^p \, dx \leqslant C^p \omega^p(f, n^{-1})_{L^p}.$$

Proof. Obviously,

$$f(x)-\nabla_n(f, x)=\sum_{k=1}^{n} nr_k(x) \int_{(k-1)/n}^{k/n} (f(x)-f(u)) \, du.$$

By Lemma 1, $r_k(x) \leqslant C \exp(-M(n-k))$ for $x \in [1-1/n, 1]$ and all $k=1, 2, ..., n$. First assume $1<p<\infty$. By using Hölder's inequality, we calculate

$$\begin{aligned}
&\int_{1-1/n}^{1} |f(x)-\nabla_n(f, x)|^p \, dx \\
&\quad = \int_{1-1/n}^{1} \left| \sum_{k=1}^{n} nr_k(x) \int_{(k-1)/n}^{k/n} |f(x)-f(u)| \, du \right|^p dx \\
&\quad \leqslant n^p \int_{1-1/n}^{1} \left(\sum_{k=1}^{n} r_k^{p/(2p-2)}(x) \right)^{p-1} \\
&\qquad \times \sum_{k=1}^{n} r_k^{p/2}(x) \left| \int_{(k-1)/n}^{k/n} |f(x)-f(u)| \, du \right|^p dx. \qquad (2)
\end{aligned}$$

From Lemma 1,

$$\begin{aligned}
\sum_{k=1}^{n} r_k^{p/(2p-2)}(x) &\leqslant C^{p/(2p-2)} \sum_{k=1}^{n} \exp(-pM(n-k)/(2p-2)) \\
&\leqslant C^{p/(2p-2)} \left(\sum_{v=1}^{\infty} e^{-Mv/2} \right)^{p/(p-1)} \leqslant C^{p/(p-1)},
\end{aligned}$$

then (2) yields that (by Hölder's inequality again)

$$\begin{aligned}
&\int_{1-1/n}^{1} |f(x)-\nabla_n(f, x)|^p \, dx \\
&\quad \leqslant C^p n \int_{1-1/n}^{1} \sum_{k=1}^{n} r_k^{p/2}(x) \int_{(k-1)/n}^{k/n} |f(x)-f(u)|^p \, du \, dx. \qquad (3)
\end{aligned}$$

When $p=1$, a direct (and much simpler) calculation can be applied to get the desired inequality (3). That is, for $1 \leqslant p<\infty$, (3) always holds.

Applying Lemma 1 and Lemma 2 to (3), we get

$$
\begin{aligned}
&\int_{1-1/n}^{1} |f(x)-\nabla_n(f,x)|^p\,dx \\
&\quad \leqslant C^p n \int_{1-1/n}^{1} \sum_{k=1}^{n} e^{-pM(n-k)/2} \int_{(k-1)/n}^{k/n} |f(x)-f(u)|^p\,du\,dx \\
&\quad = C^p n \sum_{k=1}^{n} e^{-pM(n-k)/2} \int_{(k-1)/n}^{k/n} \int_{1-1/n}^{1} |f(x)-f(u)|^p\,du\,dx \\
&\quad \leqslant C^p n \sum_{k=1}^{n} \frac{n-k+1}{n}\, e^{-pM(n-k)/2} \omega^p(f,(n-k+1)/n)_{L^p}
\end{aligned}
$$

(Note that $(k-1)/n = 1-(n-k+1)/n$)

$$
\begin{aligned}
&\leqslant C^p \omega^p(f,1/n)_{L^p} \sum_{j=1}^{\infty} j^{p+1} e^{-pMj/2} \\
&\leqslant C^p \omega^p(f,1/n)_{L^p} \left(\sum_{j=1}^{\infty} j^2 e^{-Mj/2}\right)^p \\
&\leqslant C^p \omega^p(f,1/n)_{L^p},
\end{aligned}
$$

therefore we are done. ∎

The last lemma relates the Steklov functions, and its proof is elementary and straightforward.

Lemma 4. *Let* $h>0$, $0\leqslant x\leqslant 1-h$,

$$f_h(x) = h^{-1}\int_0^h f(x+t)\,dt.$$

Then

$$\int_0^{1-1/n} |f(x)-f_h(x)|^p\,dx \leqslant \omega(f,h)_{L^p},$$

and

$$f'_h(x) = h^{-1}(f(x+h)-f(x)).$$

3. PROOF OF THEOREM 2

For $f \in L^p_{[0,1]}$, define $\nabla_n(f, x)$ as in (1). Take $h = 1/n$, then[4]

$$\int_0^{1-1/n} |f_h(x) - \nabla_n(f, x)|^p \, dx$$

$$= \int_0^{1-1/n} \left| \sum_{k=1}^n r_k(x) \int_x^{x_{k-1}} f'_h(u)\, du \right|^p \qquad \text{(by Lemma 4)}$$

$$= \sum_{j=1}^{n-1} \int_{(j-1)/n}^{j/n} \left| \sum_{k=1}^n r_k(x) \int_x^{x_{k-1}} f'_h(u)\, du \right|^p dx.$$

In a similar way to the proof of Lemma 3, by using Hölder's inequality repeatedly, we have

$$\int_0^{1-1/n} |f_h(x) - \nabla_n(f, x)|^p \, dx$$

$$\leqslant C^p \sum_{j=1}^{n-1} \int_{(j-1)/n}^{j/n} \sum_{k=1}^n r_k^{p/2}(x)\, |x - x_{k-1}|^{p-1} \times \left| \int_x^{x_{k-1}} |f'_h(u)|^p \, du \right| dx$$

$$\leqslant C^p n^{-p+1} \sum_{j=1}^{n-1} \sum_{k=1}^n |k-j|^{p-1} \int_{(j-1)/n}^{j/n} r_k^{p/2}(x) \times \left| \int_x^{x_{k-1}} |f'_h(u)|^p \, du \right| dx$$

$$= C^p n^{-p+1} \sum_{j=1}^{n-1} \sum_{k=1}^n |k-j|^{p-1} \times \left| \int_0^{1/n} r_k^{p/2}(x + (j-1)/n) \int_{x+(j-1)/n}^{x_{k-1}} |f'_h(u)|^p \, du\, dx \right|$$

$$\leqslant C^p n^{-p+1} \sum_{j=1}^{n-1} \sum_{k=1}^n |k-j|^{p-1} \times \left| \int_0^{1/n} r_k^{p/2}(x + (j-1)/n) \int_{x_j^*}^{x_{k-1}} |f'_h(u)|^p \, du\, dx \right|, \qquad (4)$$

where we take $x_j^* = (j-1)/n$ when $(j-1)/n < x_{k-1} = (k-1)/n$, and take $x_j^* = j/n$ when $(j-1)/n \geqslant x_{k-1}$, or in other words, in any case, we extend

[4] $n \int_{(k-1)/n}^{k/n} f(u)\, du = f_h(x_{k-1})$ should be noticed!

the integration limits. We continue to do the calculation by exchanging both the integrations and the summations to get

$$\sum_{j=1}^{n-1} \sum_{k=1}^{n} |k-j|^{p-1} \left| \int_0^{1/n} r_k^{p/2}(x+(j-1)/n) \int_{x_j^*}^{x_{k-1}} |f'_h(u)|^p \, du \, dx \right|$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n-1} |k-j|^{p-1}$$

$$\times \left| \int_{x_j^*}^{x_{k-1}} |f'_h(u)|^p \int_0^{1/n} r_k^{p/2}(x+(j-1)/n) \, dx \, du \right|$$

$$\leqslant C^{p/2} n^{-1} \sum_{k=1}^{n} \sum_{j=1}^{n-1} |k-j|^{p-1} \exp(-pM\,|k-j|/2)$$

$$\times \left| \int_{x_j^*}^{x_{k-1}} |f'_h(u)|^p \, du \right| \qquad \text{(by Lemma 1).}$$

Together with (4), we have now

$$\int_0^{1-1/n} |f_h(x) - \nabla_n(f, x)|^p \, dx$$

$$\leqslant C^p n^{-p} \sum_{k=1}^{n} \sum_{j=1}^{n-1} |k-j|^{p-1} \exp(-pM\,|k-j|/2)$$

$$\times \left| \int_{x_j^*}^{x_{k-1}} |f'_h(u)|^p \, du \right|. \tag{5}$$

It is not difficult to deduce the following estimate

$$\sum_{k=1}^{n} \sum_{j=1}^{n-1} |k-j|^{p-1} \exp(-pM\,|k-j|/2) \left| \int_{x_j^*}^{x_{k-1}} |f'_h(u)|^p \, du \right|$$

$$= \sum_{m=1}^{n-1} \sum_{0 \leqslant j,\, k \leqslant n;\ |k-j| = m} |k-j|^{p-1} \exp(-pM\,|k-j|/2)$$

$$\times \left| \int_{x_j^*}^{x_{k-1}} |f'_h(u)|^p \, du \right|$$

$$\leqslant C^p \sum_{m=1}^{n-1} m^{p-1} e^{-pMm/2}\, 2m \int_0^{1-1/n} |f'_h(u)|^p \, du$$

$$\leqslant C^p \sum_{m=1}^{n-1} m^p e^{-pMm/2} \int_0^{1-1/n} |f'_h(u)|^p \, du$$

$$\leqslant C^p \left(\sum_{m=1}^{\infty} m e^{-Mm/2} \right)^p \int_0^{1-1/n} |f'_h(u)|^p \, du$$

$$\leqslant C^p \int_0^{1-1/n} |f'_h(u)|^p \, du. \tag{6}$$

From Lemma 4, we see that

$$\int_0^{1-1/n} |f'_h(u)|^p \, du = h^{-p} \int_0^{1-1/n} |f(u+h) - f(u)|^p \, du$$

$$\leqslant n^p \omega^p(f, n^{-1})_{L^p}. \tag{7}$$

Combining (5)–(7) with Lemma 4, we then have

$$\int_0^{1-1/n} |f(x) - \nabla_n(f, x)|^p \, dx$$

$$\leqslant 2^p \left(\int_0^{1-1/n} |f(x) - f_h(x)|^p \, dx + \int_0^{1-1/n} |f_h(x) - \nabla_n(f, x)|^p \, dx \right)$$

$$\leqslant C^p \omega^p(f, n^{-1})_{L^p};$$

hence, with Lemma 3, Theorem 2 is completed. ∎

Remark. For positive linear polynomial operators with the form $\sum_{k=1}^{n} n \times \int_{(k-1)/n}^{k/n} f(t)\, dt \times P_{n,k}(x)$ (for example, the Bernstein–Kantorovich operators), to consider the approximation in L^p spaces people usually use some local asymptotic formulae of $P_{n,k}(x)$ to achieve the required estimates (this kind of work started from Totik [4], etc.). That method usually cannot be applied to rational operators, especially to our case. Also, because maximum functions are used, the constant in Jackson type estimates in polynomial operator approximation depends upon p for $1 < p \leqslant \infty$. From this point of view, the method used in this paper could be very useful in estimating the approximation rate by rational operators in L^p spaces.

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